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RECONSTRUCTING MATHEMATICS PEDAGOGY FROM A CONSTRUCTIVIST PERSPECTIVE

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Constructivist theory has been prominent in recent research on mathematics learning and has provided a basis for recent mathematics education reform efforts. Although constructivism has the potential to inform changes in mathematics teaching, it offers no particular vision of how mathematics should be taught; models of teaching based on constructivism are needed. Data are presented from a whole-class, constructivist teaching experiment in which problems of teaching practice required the teacher/researcher to explore the pedagogical implications of his theoretical (constructivist) perspectives. The analysis of the data led to the development of a model of teacher decision making with respect to mathematical tasks. Central to this model is the creative tension between the teacher's goals with regard to student learning and his responsibility to be sensitive and responsive to the mathematical thinking of the students.

Constructivist perspectives on learning have been central to much of recent empirical and theoretical work in mathematics education (Steffe & Gale, 1995; von Glasersfeld, 1991) and as a result, have contributed to shaping mathematics reform efforts (National Council of Teachers of Mathematics, 1989, 1991). Although constructivism has provided mathematics educators with useful ways to understand learning and learners, the task of reconstructing mathematics pedagogy on the basis of a constructivist view of learning is a considerable challenge, one that the mathematics education community has only begun to tackle. Although constructivism provides a useful framework for thinking about mathematics learning in classrooms and therefore can contribute in important ways to the effort to reform classroom mathematics teaching, it does not tell us how to teach mathematics; that is, it does not stipulate a particular model.

The word "pedagogy," as used above, is meant to signify all contributions to the mathematical education of students in mathematics classrooms. As such, it includes not only the multi-faceted work of the teacher but also the contributions to classroom learning of curriculum designers, educational materials developers, and educational researchers. Mathematics pedagogy might be operationally defined using the following thought experiment. Picture 25 learners in an otherwise empty classroom.

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The ingredient necessary in order to initiate mathematics learning is *pedagogy*.

This paper describes data from a classroom teaching experiment in which the researcher served as mathematics teacher, the analysis of that data, and an emerging theoretical framework for mathematics pedagogy that derives from the analysis. The paper contributes to a dialogue on *what teaching might be like if it were built on a constructivist view of knowledge development*. The specific focus of this paper is on decision making with respect to the mathematics content and mathematical tasks for classroom learning.

This article begins with an articulation of the constructivist perspective that undergirds the research and teaching and then provides a review of the pedagogical theory development based on constructivism that preceded this study and contributed to its theoretical foundation. The study reported here examines the pedagogical decisions that result from the accommodation of the researcher's theoretical perspectives to the problems of teaching.

A CONSTRUCTIVIST PERSPECTIVE

The widespread interest in constructivism among mathematics education theorists, researchers, and practitioners has led to a plethora of different meanings for "constructivism." Although terms such as "radical constructivism" and "social constructivism" provide some orientation, there is a diversity of epistemological perspectives even within these categories (cf. Steffe & Gale, 1995). Therefore, it seems important to describe briefly the constructivist perspective on which our research is based.

Constructivism derives from a philosophical position that we as human beings have no access to an objective reality, that is, a reality independent of our way of knowing it. Rather, we construct our knowledge of our world from our perceptions and experiences, which are themselves mediated through our previous knowledge. Learning is the process by which human beings adapt to their experiential world.

From a constructivist perspective, we have no way of knowing whether a concept matches an objective reality. Our concern is whether it *works* (fits with our experiential world). Von Glasersfeld (1987, 1995) refers to this as "viability," in keeping with the biological model of learning as adaptation developed by Piaget (1970). To clarify, a concept works or is viable to the extent that it does what we need it to do: to make sense of our perceptions or data, to make an accurate prediction, to solve a problem, or to accomplish a personal goal. Confrey (1995) points out that a corollary to the radical constructivist epistemology is its "recursive fidelity—constructivism is subject to its own claims about the limits of knowledge. Thus, [constructivism] is only true to the extent that it is shown useful in allowing us to make sense of our experience." When what we experience differs from the expected or intended, disequilibrium results and our adaptive (learning) process is triggered. Reflection on successful adaptive operations (reflective abstraction) leads to new or modified concepts.

Perhaps the most divisive issue in recent epistemological debates (Steffe & Gale, 1995) is whether knowledge development (particularly relational knowledge) is seen as fundamentally a social process or a cognitive process. The difference in the two positions seems to depend on the focus of the observer. The radical constructivist position focuses on the individual's construction, thus taking a cognitive or psychological perspective. Although social interaction is seen as an important context for learning, the focus is on the resulting reorganization of individual cognition.

> For Piaget, just as for the contemporary radical constructivist, the "others" with whom social interaction takes place, are part of the *environment*, no more but also no less than any of the relatively "permanent" objects the child constructs within the range of its lived experience. (von Glasersfeld, 1995)

On the other hand, epistemologists with a sociocultural orientation see higher mental processes as socially determined. "Sociocultural processes are given analytical priority when understanding individual mental functioning rather than the other way around." (Wertsch & Toma, 1995) From a social perspective, knowledge resides in the culture, which is a system that is greater than the sum of its parts.

Our position eschews either extreme and builds on the theoretical work of Cobb, Yackel, and Wood (Cobb, 1989; Cobb, Yackel, & Wood, 1993; Wood, Cobb, & Yackel, 1995) and Bauersfeld (1995), whose theories are grounded in both radical constructivism (von Glasersfeld, 1991) and symbolic interactionism (Blumer, 1969). Cobb (1989) points out that the coordination of the two perspectives is necessary to understand learning in the classroom. The issue is not whether the social or cognitive dimension is primary, but rather what can be learned from combining analyses from these two perspectives. I draw an analogy with physicists' theories of light. Neither a particle theory nor a wave theory of light is sufficient to characterize the physicist's data. However, it has been useful to physicists to consider light to be a particle and to consider light to be a wave. Coordinating the findings that derive from each perspective has led to advancements in the field. Likewise, it seems useful to coordinate analyses on the basis of psychological (cognitive) and sociological perspectives in order to understand knowledge development in classrooms.

Psychological analysis of mathematics classroom learning focuses on individuals' knowledge of and about mathematics[1], their understanding of the mathematics of the others, and their sense of the functioning of the mathematics class. Sociological analysis focuses on taken-as-shared knowledge and classroom social norms (Cobb, Yackel, & Wood, 1989). "Taken-as-shared" (Cobb, Yackel, & Wood, 1992; Streeck, 1979) indicates that members of the classroom community, having no direct access to each other's understanding, achieve a sense that some aspects of knowledge are shared but have no way of knowing whether the ideas are in fact shared. "Social norms" refer to that which is understood by the community as constituting effective participation in the mathematics classroom community. The social norms

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[1]Ball (1991) defines knowledge *of* mathematics as conceptual and procedural knowledge of the subject and knowledge *about* mathematics as "understandings about the nature of mathematical knowledge and activity: what is entailed in doing mathematics and how truth is established in the domain. What counts as a solution in mathematics? How are solutions justified and conjectures disproved? Which ideas are arbitrary or conventional and which are necessary or logical?" (p. 7)

include the expectations that community members have of the teacher and students, the conceptions of what it means to do mathematics in that community, and the ways that mathematical validity is established.

> It is useful to see mathematics as both cognitive activity constrained by social and cultural processes, and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals (Wood, Cobb, & Yackel, 1995).

We refer to this coordination of psychological and sociological analyses as "social constructivism."

SOCIAL CONSTRUCTIVISM AND MATHEMATICS PEDAGOGY

Understanding learning as a process of individual and social construction gives teachers a conceptual framework with which to understand the learning of their students. Although the development of such understandings is extremely valuable, this paper focuses on the question of how constructivism might contribute to a reconstruction of mathematics pedagogy. How might it inform the development of a framework for fostering and supporting learners' constructions of powerful ideas? Wood, Cobb, and Yackel (1995) assert that

> teachers must ... construct a form of practice that fits with their students' ways of learning mathematics. This is the fundamental challenge that faces mathematics teacher educators. We have to reconstruct what it means to know and do mathematics in school and thus what it means to teach mathematics.

As I stated above, constructivism, as an epistemological theory, does not define a particular way of teaching. It describes knowledge development whether or not there is a teacher present or teaching is going on. Konold (1995) argues, "not that a teacher's epistemology has *no* effect on how he or she teaches, rather that its effects are neither straightforward nor deterministic." There is no simple function that maps teaching methodology onto constructivist principles. A constructivist epistemology does not determine the appropriateness or inappropriateness of teaching strategies. Bauersfeld (1995) states,

> The fundamentally constructive nature of human cognition and the processual emergence of themes, regularities, and norms for mathematizing across social interaction, to bring the [psychological] and the social together, make it impossible to end up with a simple prescriptive summary for teaching. There is no way towards an operationalization of the social constructivist perspective without destroying the perspective.

The commonly used misnomer, "constructivist teaching," however, suggests to the contrary that constructivism offers one set notion of how to teach. The question of whether teaching is "constructivist" is not a useful one and diverts attention from the more important question of how effective it is. From a theoretical perspective, the question that needs attention is, In what ways can constructivism contribute to the development of useful theoretical frameworks for mathematics pedagogy?

It is overly simplistic and not useful to connect constructivism to teaching with the romantic notion, "Leave students alone and they will construct mathematical understandings." Likewise, "Put students in groups and let them communicate as they solve problems," is not much more helpful. History provides unsolicited empirical evidence with respect to these approaches. Generations of outstanding mathematicians who were engaged in mathematical problems, who communicated with their colleagues about their work, required thousands of years to develop mathematics that we expect our average elementary school students to construct (Richards, 1991). Thus, although it is useful to have students work problems and communicate about their ideas, it does not seem to be adequate as a prescription for mathematics teaching. The challenge is, *How can mathematics teachers foster students' construction of powerful mathematical ideas that took the community of mathematicians thousands of years to develop?*

Richards asserts,

> It is necessary [for the mathematics teacher] to provide a structure and a set of plans that support the development of *informed* exploration and *reflective* inquiry without taking initiative or control away from the student. The teacher must design tasks and projects that stimulate students to ask questions, pose problems, and set goals. Students will not become active learners by accident, but *by design*, through the use of the plans that we structure to guide exploration and inquiry.[2] (p. 38 [Italics in the original])

Through empirical data and model building, this study attempts to examine the process of constituting pedagogical designs.

RECENT THEORETICAL WORK ON PEDAGOGICAL FRAMEWORKS

Relatively little work in mathematics education has focused on the development of theoretical frameworks for mathematics pedagogy consistent with constructivism. This seems to be the result of several factors:

1. It is only recently that empirically based models for studying mathematics learning in classrooms have been articulated (cf. Wood, Cobb, Yackel, & Dillon, 1993). Earlier empirical work, which derived from, and contributed to, epistemological theory, focused on the cognitive development of individual learners (cf. Steffe, von Glasersfeld, Richards, & Cobb, 1983).

2. Traditional views of mathematics, learning, and teaching have been so widespread that researchers studying teachers' thinking, beliefs, and decision making have had little access to teachers who had well-developed constructivist perspectives and who understood and were implementing current reform ideas. As a result there has been a lack of connection between research on learning (which has focused on constructivism) and research on teaching (which has focused for the most part on traditional instruction).

3. The need for pedagogical frameworks is sometimes obscured by the tendency to assume that constructivism defines an approach to teaching.

[2] I interpret Richards's statement, "Students will not become active learners..." as indicative of his interest in fostering more independent and reflective mathematical investigations and discussions among students. From a constructivist perspective, students are always active learners; however, the nature of what is constructed in different classroom contexts may vary greatly.

Despite these factors, some important work has been done in recent years with respect to rethinking mathematics teaching on the basis of a constructivist perspective (in some cases without specific reference to constructivism). This work has focused on identifying the roles of mathematics teachers and describing the nature of "pedagogical deliberations" (Ball, 1993).

The *Professional Standards for School Mathematics* (National Council of Teachers of Mathematics, 1991) envisions teachers' responsibilities in four key areas:

- Setting goals and selecting or creating mathematical *tasks* to help students achieve these goals;
- Stimulating and managing classroom *discourse* so that both the students and the teacher are clearer about what is being learned;
- Creating a classroom *environment* to support teaching and learning mathematics;
- *Analyzing* student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions. (p. 5)

Cobb, Wood, and Yackel (1993) elaborate the teacher's responsibilities in the mathematics classroom. The teacher has the dual role of fostering the development of conceptual knowledge among her or his students and of facilitating the constitution of shared knowledge in the classroom community. Cobb et al. (1993) have demonstrated that classroom conversations about mathematics, facilitated by the teacher, result in taken-as-shared mathematical knowledge. They have also described a second type of conversation that focuses on what constitutes appropriate and effective mathematical activity in the classroom. Such discussion contributes to the constitution and modification of social norms for mathematical activity, the *contrat didactique*[3] (Brousseau, 1981).

Much of the teacher's responsibilities involve planning. However, the planning of instruction based on a constructivist view of learning faces an inherent tension. Brousseau emphasizes that students must have freedom to make a response to a *situation* on the basis of their past knowledge of the context and their developing mathematical understandings. If the situation *leads* the students to a particular response, no real learning of the mathematical ideas underlying that response takes place. However, "if the teacher has no intention, no plan, no problem or well-developed situation, the child will not do and will not learn anything" (Brousseau, 1987, p. 8—my translation). Under these conditions, students learn other things, such as how to respond appropriately to the teacher's leading questions.

Brousseau (1983), Douady (1985), Lampert (1990), and Ball (1993) have conducted investigations into the nature of pedagogical thinking and decision making that contribute to teacher planning. Brousseau (1987) asserts that part of the role of the teacher is to take the noncontextualized mathematical ideas that are to be taught and embed them in a context for student investigation. Such a context should be personally meaningful to the students, allowing them to solve problems in that context, the solution of which might be a specific instantiation of the idea to be learned. (Ball's [1993]

[3]The *contrat didactique* is also established by classroom routines that are not explicitly discussed.

notion of "representational context" seems consistent with Brousseau's *situation*.) The teacher's job is to propose a learning situation within which students seek a response to the *milieu*, not a response that is solely intended to please the teacher. For the problem to foster the learning of powerful mathematical ideas, the students must accept the problem as *their* problem[4]; they must accept the responsibility for truth (Balacheff, 1990). Brousseau calls this the *devolution* of the problem.

The creation of appropriate problem contexts (*situations a-didactiques*) is not sufficient for learning. Brousseau points out that situations must be created for the decontextualizing and depersonalizing of the ideas (*situations didactiques*). Learning involves being able to use the ideas beyond the narrow context of the original problem situation. "The teaching process should allow for this shift of pupils' interest from being practitioners to becoming theoreticians" (Balacheff, 1990, p. 264).

Also necessary is what the French researchers call "situations for institutionalization" (Brousseau, 1987; Douady, 1985), in which ideas constructed or modified during problem solving attain the status of knowledge in the classroom community. This is consistent with the notion of mathematical knowledge as social knowledge, as knowledge that is taken-as-shared by the classroom community.

Lampert's (1990) use of "problems" corresponds with Brousseau's situations and Ball's representational contexts. Lampert describes the pedagogical thinking in which she engages to generate problems for her students.

> At the beginning of a unit, when we were switching to a new topic, the problem we started with was chosen for its potential to expose a wide range of students' thinking about a bit of mathematics, to make explicit and public what they could do and how they understand. Later problems were chosen based on an assessment of the results of the first and subsequent discussions of a topic, moving the agenda along into new but related mathematical territory. The most important criterion in picking a problem was that it be the sort of problem that would have the capacity to engage all of the students in the class in making and testing mathematical hypotheses. These hypotheses are imbedded in the answers students give to the problem, and so comparing answers engaged the class in a discussion of the relative mathematical merits of various hypotheses, setting the stage for the kind of zig-zag between inductive observation and deductive generalization that Lakatos and Polya see as characteristic of mathematical activity. (p. 39)

Such pedagogical thinking must be built on knowledge of mathematics and knowledge of students and how they learn mathematics. Ball (1993) points out that teachers must have a "bifocal perspective—perceiving the mathematics through the mind of the learner while perceiving the mind of the learner through the mathematics" (p.159). Steffe (1991) stresses that the teachers' plans must be informed by the "mathematics *of* students." "The most basic responsibility of constructivist teachers is to learn the mathematical knowledge of their students and how to harmonize their teaching methods with the nature of that mathematical knowledge" (Steffe & Wiegel, 1992, p. 17).

Decisions as to the nature and sequence of the mathematics to be taught are made, according to Laborde (1989), on the basis of hypotheses about epistemology

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[4]Although each problem solver may construct a somewhat different understanding of the problem, negotiation commonly takes place in the classroom to arrive at a taken-as-shared interpretation of the problem.

and learning. Ball (1993) further explicates the investigative nature of teaching: "Teaching is essentially an ongoing inquiry into content and learners and into ways that contexts can be structured to facilitate the development of learners' understandings" (p. 166).

Ball states that research is needed to further understand the pedagogical deliberations in reform-oriented mathematics teaching. Building on the work of the researchers cited above and starting from a social constructivist perspective on knowledge development, my paper continues the discussion of pedagogical deliberations that lead to the determination of problem contexts for student involvement. In particular, the paper extends the notion of teaching as inquiry, examines the role of different aspects of teachers' knowledge, and explores the ongoing and inherent challenge to *integrate the teacher's goals and direction for learning with the trajectory of students' mathematical thinking and learning.*

THEORY MEETS PRACTICE IN THE CONTEXT OF A TEACHING EXPERIMENT

This section focuses on data from a classroom teaching experiment, in order to analyze situations in which a constructivist theoretical perspective came up against the realities of real students in a real classroom. The nonroutine problems of teaching require an elaboration and modification of theories of learning and teaching. When the researcher/theorist assumes the role of teacher in a research project, he is uniquely positioned to study in a direct way the interaction of his theory and practice. Particularly, this report focuses on the teacher/researcher's ongoing decision making with respect to the mathematical content of the course and the tasks and questions that provided a context for the study of that content. This section begins with some brief background on the teaching experiment.

Background

The teaching experiment was part of the Construction of Elementary Mathematics (CEM) Project, a 3-year study of the mathematical and pedagogical development of prospective elementary teachers. The project studied the prospective teachers in the context of an experimental teacher preparation program designed to increase their mathematical knowledge and to foster their development of views of mathematics, learning, and teaching that were consistent with the views espoused in recent reform documents (e.g., National Council of Teachers of Mathematics, 1989; 1991). Data collection with 26 prospective elementary teachers (20 of whom finished the program) proceeded throughout a mathematics course, a course on mathematics learning and teaching, a 5-week pre–student-teaching practicum, and a 15-week student-teaching practicum.

The research on the mathematics course and the course on mathematics learning and teaching employed a constructivist teaching-experiment methodology, as described by Cobb and Steffe (1983) for research with individual subjects. We adapted that methodology to research on classroom mathematics (in the manner of Cobb et al., 1993). The author taught all classes. Classes were videotaped and field notes were taken by project researchers. Videotapes of classes were transcribed for analysis. The author kept a reflective notebook in which he recorded his thinking immediately following teaching and planning sessions. Following each class the author met with a second project researcher to discuss what he and his colleague inferred the conceptualizations of the students to be at that point and to plan for the next instructional intervention. (In this section, "students" refers to the prospective elementary teachers participating in the teaching experiment.) These meetings were audiotaped.

The teaching-experiment methodology involves "hypothesizing what the [learner] might learn and finding ways of fostering this learning" (Steffe, 1991, p. 177). This research report represents an extension of the teaching-experiment methodology. Whereas the teaching experiment was created to learn about students' developing conceptions (our primary emphasis), analysis of the decision making of the teacher/researcher in posing problems is potentially a rich source for learning about teaching (Cobb, personal communication). This paper is based on such an analysis.

Class lessons generally consisted of small-group problem solving and teacher-led whole-class discussions. No lectures were given. The primary mathematical goal of the course was for students to learn to identify multiplicative relationships (Simon & Blume, 1994a, 1994b). Previous research on a variety of populations (Hart, 1981; Inhelder & Piaget, 1958; Karplus, Karplus, Formisano, & Paulson, 1979) and our pretest data with this population of students had shown that identifying ratio relationships tends to be difficult and that additive comparisons are often used where multiplicative comparisons (ratios) are more appropriate. The mathematical content of the course began with exploration of the multiplicative relationship involved in evaluating the area of rectangles.

Data and Analysis

The data presented focuses on three teaching situations, examining the relationship among the teacher's decision making and the classroom activities. These data are from the first 5 weeks of the 15-week mathematics course, the duration of the first instructional unit, and are taken from class transcripts, the teacher/researcher's notes, field notes from other researchers, and student journals. The first unit (eight 90-minute classes) focused on understanding the multiplicative relationship involved in evaluating the area of a rectangle. The three instructional situations described represent the three subtopics of the instructional unit. For each situation, a description is provided of the challenge that faced the teacher as construed by the teacher, the decision that he made to respond to that challenge, and the subsequent classroom interaction that was constituted by the students and teacher.

Note that fundamental to the teacher's understanding of the challenge was his constructivist perspective, which included the idea that students construct their understandings, they do not absorb the understandings of their teachers. Each of the three situations represents an attempt to promote and support powerful constructions. Whereas telling students what they should understand (a lecture approach) is relatively straightforward, developing *situations a-didactiques* or representational contexts is complex and uncertain. In this latter approach, mathematics teaching is continually problematic.

The rectangles problem. As the instructor, I chose to begin the exploration of multiplicative relationships in the evaluation of area of rectangles. My purpose was to focus on the multiplicative relationships involved, *not* to teach about area. The lesson I chose was one that I had used several times before with similar groups of students. The lesson grew out of my observation that although many prospective elementary teachers respond to area-of-rectangle problems by multiplying, their choice of multiplication is often the result of having learned a procedure or formula rather than the result of a solid conceptual link between their understandings of multiplication and their understandings of measuring area. This lesson, which was designed to foster the development of that link, was planned to be completed in one day, although I anticipated that it might continue into the next class.

The lesson began with a small cardboard rectangle being given to each of the small groups of students seated at the classroom's six rectangular tables. The groups were challenged to solve the following problem.

> *Rectangles problem 1.* Determine how many rectangles, of the size and shape of the rectangle that you were given, could fit on the top surface of your table. Rectangles cannot be overlapped, cannot be cut, nor can they overlap the edges of the table. Be prepared to describe to the class how you solved this problem.

Each group of students used the given rectangle as a measure to count the number of rectangles along the length of the table and the number of rectangles along the width of the table and then multiplied these two quantities. (For an extensive analysis of the quantitative reasoning involved in this instructional unit, see Simon & Blume, 1994b.[5]) However, a few of the groups raised the question of whether the orientation of the rectangle should be maintained for the second measurement (see Figure 1a), or whether it should be rotated 90 degrees so that measuring is always done using the same side of the rectangle (see Figure 1b).

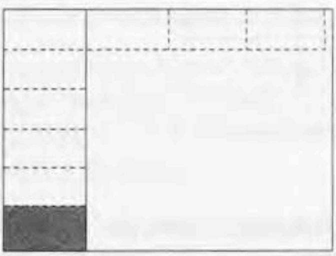


Figure 1a. Maintaining orientation of the rectangle.

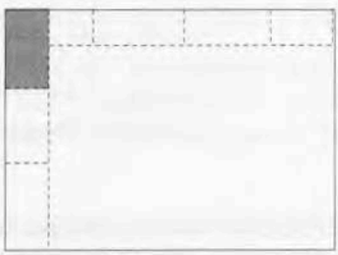


Figure 1b. Rotating the rectangle to measure the adjacent side.

During whole-class discussion, students described how they had solved the problem. Then, to focus discussion on the multiplicative relationships, I asked them

[5]Much of the data for this report was reported earlier in Simon and Blume (1994b). The earlier article focuses on the students' quantitative reasoning. This article revisits some of the data to unpack the pedagogical issues.

why they had multiplied these numbers. Some responded by saying that "it seemed like the easiest way," or "in previous math classes you learned the formula for areas" (Simon & Blume, 1994b). Others said it works; the product is the same as the result of counting up all the rectangles. I asked whether there was reason to expect that it would always work. From my perspective, it is fundamental in mathematics to consider whether a claim could be defended that the observed phenomenon would always occur under a particular set of conditions. Most of the students seemed perplexed by this question. However, Molly explained:

Molly: Well, it would work because, um, multiplying and adding are related, in that multiplying is, is like adding groups, and so it would always work because you add them up to see how many is in the square and to multiply the groups that go like that, that'll always work. You would get the same number, I'm saying if you added them or if you multiplied that side times that side. Because you're adding, I mean, you're multiplying the number of groups by the number in the groups, which is the same as adding them all up.

Molly clarified her explanation by demonstrating on the chalkboard how each row of rectangles was a group (see Figure 2) and the number of rectangles in a row was the number in each group. She showed that summing the rectangles in each row (repeated addition) was equivalent to multiplying the number in a row by the number of rows (Simon & Blume, 1994b).

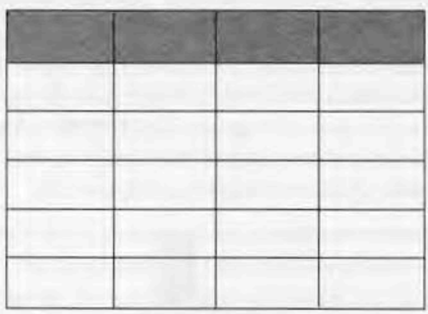


Figure 2. One group of four rectangles.

The other students' subsequent comments suggested that only a few of them perceived that Molly's explanation had advanced the discussion in any significant way.

Situation 1. What instructional situation might afford other students the opportunity to construct understandings similar to Molly's? It wasn't that the other students were puzzled by Molly's explanation; they seemed unaffected by it. They continued to respond to the question, "Why multiply?" in ways that indicated that the question did not demand for them this type of justification. Responses included "...'cause that's the way we've been taught." and "... it's a mathematical law." Asking the students for explanations and justifications was not sufficient. Our classroom community had not established what counts for mathematical justification (Simon & Blume, in press). It did not seem that continuing this already lengthy discussion would be fruitful.

I had engaged them in a problem-solving activity using a hands-on activity and fostered communication in small and large groups. Yet only a few of my students showed evidence of learning the mathematics that I had intended them to learn. The challenge that I faced was a product of both cognitive and social factors. Cognitively, the majority of my students were employing a procedure that was well-practiced but not well-examined conceptually. Socially, they did not have a view of mathematical activity in general and of appropriate activity for our classroom in particular that included the type of relational thinking and development of justification in which I was attempting to engage them.

I realized that the development of norms for classroom mathematics activity would take some time. Such norms would result from the activities in which we engaged as a mathematical community and the discussions that we had about that activity. Their competence in providing justification would grow as they engaged in discussions in which the demand for justification was consistently present (Simon & Blume, in press). Thus, from a social perspective, I needed to continue the process that I had begun with them. However, this process could not happen in the abstract. Particular content and tasks were needed as the context for the constituting of appropriate mathematical activity. Thus, I returned to my role of problem poser, but the question was, "Which problems?" The traditional approach—assigning practice problems similar to the original one—seemed inappropriate. After all, the students were already able to generate correct answers; the real problem was understanding why multiplication was appropriate. I needed to find problems that necessitated an understanding of the link between the solution strategy, counting the number of rectangles along the length and the width and multiplying those quantities, and the goal of determining the total number of rectangles that could be laid out on the table.

To generate such problems, I made use of conceptual difficulties that I had previously observed among students working on Rectangles Problem 1. For example:

> *Rectangles problem 2*. Bill said, "If the table is 13 rectangles long and 9 rectangles wide, and if I count 1, 2, 3 …, 13 and then again 1, 2, 3 …, 9, and then I multiply, 13 × 9, then I have counted the corner rectangle twice." Respond to Bill's comment.

Problem 2 seemed to engage students in making a conceptual link between the goal of counting all of the rectangles and the prevalent solution strategy of counting rectangles along the two sides and multiplying. The following excerpts from the class transcript show the development of these connections. (Note: "Simon" refers to me, the teacher.)

Karen: When we're multiplying thirteen times nine we're trying to see … how many nines there are.… So if I'm looking at one nine, two nines, three nines, four nines, I could find out how much it would be with those numbers, but if I'm looking for thirteen nines, I would want to see how many of those I would have if I would add them up or if I would multiply them thirteen times. How many—how much would that be if I had thirteen nines or nine thirteens? I'm looking for the amount—total amount that that would be if I was multiplying how many groups of those or how many sets of those would I have, if I would add each one of them up to get the total amount.

Karen has made some progress in justifying the use of multiplication. However, Toni goes back to how one uses the formula appropriately. Her explanation is based on her identification of the problem as an area problem and her knowledge of how to measure length times width. Once again I attempt to refocus Toni (and, I suspect, other students) on the underlying conceptual issue.

Toni: When we're trying to find how many rectangles would fill that rectangle, we're looking for the area. And when we find area we multiply length times width, and the columns would probably represent the length and the rows would represent the width....

Simon: Why does that work, that when we multiply the number of columns times the number of rows we get the area?

Molly: Well, I thought again it referred back to when you're using a row to represent the units in a group, and the columns to represent the number of groups, and since multiplication is the same as repeated addition, that when you multiplied the number of units in a group by the number of groups, you would get the total number of parts in the whole.

Simon: And how is that connected to this issue about the corner?

Molly: Because it ... the corner not only represents a one, it's just one numbering of a group, or it's also numbering a part of that unit—a unit in that group—so it's not, it's two different things, just like when they were saying it's a row and a column, well, it's two different things, it's a unit and also representing a group.

Candy: ...It makes it confusing to try to look at the length times the width.... You should really treat it as so many sets or so many groups, like nine groups ..., thirteen groups of nine. That way, you're not even going to deal with the corner and you won't even have that problem.

Karen: Have we responded to Bill's problem about him thinking that he has double-counted?

At this point, Karen brings us back to the original problem. For her, it is not enough to decide whether one would be double-counting the corner rectangle; it is also important to understand Bill's thinking, which led to his confusion.

Karen: It appears to me that Bill's thinking about ... counting by ones.... one represents two different things, but in his mind, at least from what he said, it appears that he only sees one as representing one thing, and that is a counting number. He thinks he's already counted it.

Candy's and Karen's comments seem to demonstrate an understanding of how the counting (vertically and horizontally) and the multiplication are related. I push for further verbalization of the ideas involved to ascertain whether others in the class have constructed similar meanings. Many now insist that we are really counting rows and columns. I suspect that some of the students have latched onto the notion of rows and columns in an unexamined way. The shift from counting boxes to counting rows and columns does not in itself lead to a connection between counting the total number and the multiplicative approach. I refocus my questions on this connection.

Simon: So I'm not counting boxes at all?

Class: No.

Simon: OK. Isn't it a little mysterious that we're never counting boxes here and we wind up with a number of boxes? Does that bother anybody? We didn't count boxes here, we didn't count boxes there, and we wind up with boxes at the end. Tammy?

Tammy: If each box represents a portion of the row so we're really counting boxes but we're just putting them in a set, instead of individually.

Simon: OK, so which way are you thinking about the sets going? [Points at diagram on the board] This way or that way? Choose one.

Tammy: Vertically?

Simon: So this is a set? OK. So you are saying this is nine what?

Tammy: Nine separate units inside of a set.

Simon: …OK. So here I counted nine boxes in a set and then here I'm counting what?

Tammy: Thirteen.

Simon: Thirteen what?

Tammy: Boxes in a set.

[Ellen is shaking her head]

Simon: Ellen, you don't like that.

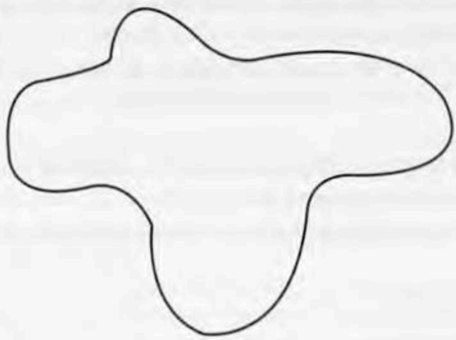
Ellen: If you're going to do it that way, I think you have to say that you're going to take the column as a set, nine boxes in one set … then the thirteen in the row is the number of sets that you have, so it's not actually boxes, it's the number of that same type of set that you have.

Ellen's final comment evoked many nods of agreement from her classmates. I considered, however, that for a student to follow an explanation might not require the same level of understanding as would be needed to generate an explanation. Still, it seemed clear to me from students' verbalizations that the number of students who were seeing a connection between multiplication and counting the total number of rectangles had increased. Perhaps Problem 2 shifted the discussion from *my* problem—justifying the method that the students believed to be valid—to a *community* problem—how to account for the "double counting." Problem 2 seemed to provide a puzzlement, at least initially for most of the students. We cannot assume, however, that all of the students related to Problem 1 as my problem, nor that all of them owned Problem 2.

Situation 2. As we proceeded to explore the multiplicative relationship involved in evaluating the area of a rectangle, I came to believe that the context in which we were working (area) was not well understood by many of the students. They seemed to think about area as generated by multiplying length times width. Although my primary focus was on multiplicative relationships, not on area, it seemed clear that an understanding of area was necessary in order for students to think about constituting the quantity (area) and evaluating that quantity. (See Simon & Blume, 1994b; and Thompson, 1994, for explications of the distinction between constitution and evaluation of quantities.) What action could I take as a teacher?

It seemed that, if indeed these students were unclear about what is meant by area, the traditional response of "reviewing" the idea would be inadequate. Surely by this time, their junior year of college, they had been present at many such reviews and had some ideas that could be built on. I chose instead to pose a problem that would push them to extend their understanding of area. I posed the following:

The *blob problem*: How can you find the area of this figure?



Students generated ideas in small groups. When we reconvened as a whole class, they shared many ideas. Two methods stimulated a great deal of discussion. The first was a suggestion that a string be put along the outline of the blob and then, without changing the length of the string, reshape it into a rectangle, a figure whose area we know how to determine. This method was based on an assumption, eventually rejected by the class, that figures that have the same perimeter also have the same area. The second method was to cut out cookie dough of a constant thickness so that it exactly covered the blob. Then, cut the cookie dough into squares of a given size. Roll up the remaining dough and roll it out to the original thickness, again cutting out squares, repeating this procedure until there is not enough dough remaining to make an additional square. This method was accepted by the class as theoretically sound; however, they predicted that in practice it would be difficult to carry out accurately.

The problem generated more than discussion on the validity of these methods. I perceived that the contrast in the methods proposed by the students would enable us to consider the issue of conservation of area, that is, under what changes in shape the original amount of area is preserved. This was a challenging context for the students to think about the meaning of area. Considerable dialogue ensued in which students seemed to be using the notion of area appropriately, comparing area among different geometric shapes, and distinguishing it from the notion of perimeter that was brought up by the string strategy.

Situation 3. Following a discussion of why multiplication was used to determine the total number of rectangles and after our work with the blob problem, I raised again the issue that they had brought up in solving the original problem, whether to turn the rectangle (Figure 1b) or to maintain its orientation (Figure 1a). I demonstrated the former, rotating it 90 degrees to measure the second side (as in Figure 1b). (The quantitative reasoning brought to bear on this problem is analyzed in depth in Simon & Blume, 1994b.)

Most of the students recognized that the method I demonstrated would not determine the number of rectangles that could fit on the table. I then asked them whether that method tells us anything about this particular table. (This question is referred to as the Turned Rectangle [TR] Problem.) Consensus developed in the class that the number generated was meaningless because the method generated a set of "overlapping rectangles."

I was surprised by their response and concerned about what it indicated about their understanding of the multiplicative (area) unit. I had posed this extension to encourage an understanding of the multiplicative relationship between linear measures of the rectangle and area measures. The measurement strategy in question and subsequent computation was identical to what they previously had learned to do with a ruler (procedurally). In prior courses, when I posed this question, several students understood that the quantity resulting from multiplying the two linear measures represented the number of *square* units of area on the table. The square unit had sides equal to the length of the rectangle used for measuring. (See Figure 3.) Generally, the discussion that ensued led to a consensus with respect to this point.

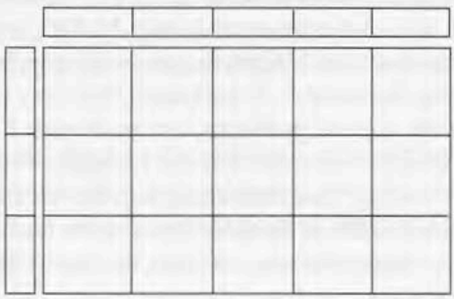


Figure 3. Constitution of square units.

However, unlike my previous experiences, no one in this class seemed to see this method as generating square units of sides equal to the length of the rectangle. Rather, they were confident in their view that the number generated by this method was nonsense because it resulted in overlapping rectangles. I tried in different ways to promote disequilibrium so the students would reconsider the issue. Toward this end, I posed the following question:

> Out in the hall I have two [rectangular] tables of different sizes. I used this method … where I measure across one way, turn the [rectangle], measure down the other way, and multiply.… When I multiplied using [this] method, on table A I got 32 as my answer and [when I measured] table B [using the same rectangle and the same method], I got 22. Now what I want to know is, [having used] the method of turning the rectangle, is table A bigger, is table B bigger, or don't you have enough information from my method to tell? (Simon & Blume, 1994b, p. 480)

The students reasoned that because 32 is greater than 22, table A must be bigger than table B. I probed, "32 what and 22 what?" They responded that the 32 and 22 did not count anything meaningful because this method created overlapping rectangles (as in the upper left-hand corner of Figure 1b).

My attempts at creating disequilibrium with my current students, a key part of my theory and practice, had been ineffectual. How could I understand the thinking of these students, and how could I work with them so that they might develop more powerful understandings? I had never encountered this pedagogical problem before. I had always posed the problem (of turning the rectangle) within the whole-class discussion, and there had always been some students who explained about the square units to their classmates. I had assumed that the other students understood.

One advantage of our teaching experiment design was that time was structured into the project to reflect, in collaboration with a colleague, on the understandings of the students. Our reflection led to the following hypotheses (developed more fully in Simon & Blume, 1994b).

As a result of Rectangles Problem 1, and because the measurement was still being done using the cardboard rectangle, the students anticipated incorrectly that the unit of area would be rectangles of the size of the cardboard one. They considered when they lay the rectangle along one edge of the table that they were making a row of rectangles, a set, an iterable unit. (The discussion of Problems 1 and 2 had encouraged students to think of a row as an iterable unit; Molly's view of a unit of units now seemed to be taken-as-shared.) When they moved the rectangle down the other side, they were counting the number of iterations. This way of viewing the situation was adequate for the original problems, but inadequate for solving the extension (TR) problem in which a unit other than the rectangle itself was being created. The students were not "seeing" that measuring with the rectangle was a process of subdividing the length and width of the table into smaller linear units and that these units together implied a rectangular array of units, the size of these units determined by the size of the linear units.

Having constructed hypotheses of the students' thinking, I still needed to generate an appropriate instructional intervention. I reasoned that if students had misanticipated the unit of area, assuming that the cardboard rectangle was the appropriate measure, then providing them with a context that did not invite misanticipation might give them the opportunity to determine an appropriate unit of area based on linear units. Eventually, they would still need to sort out the problem involving the turned rectangle. This thinking led me to generate the stick problem followed by modification of the original problem turning the rectangle problem.

> *The stick problem.* Two people work together to measure the size of a rectangular region; one measures the length and the other the width. They each use a stick to measure with. The sticks, however, are of different lengths. Louisa says, "The length is four of my sticks." Ruiz says, "The width is five of my sticks." What have they found out about the area of the rectangular region?

Students worked the stick problem in groups of three. Some of the students were able to see how the use of different size sticks could be thought of as determining an array of nonsquare rectangles. Toni began the class discussion as time was running out. Ellen picked up the discussion in the following class.

Toni: It would pretty much be the same thing as a rectangle because ... the width of the rectangle is smaller than the length of the rectangle, so it doesn't really matter, I mean, the area ... would be 20 [of these small rectangles].

Ellen: I didn't really understand until Toni drew that diagram on the board, that I found

> something out about area, and when she put that on the board, I realized that I had started thinking that you were starting with the unit, and you had to start with the unit to figure out the area. But … in my mind she sort of went backwards, she ended up with a unit of measurement by, um, making the rectangle.
>
> …Because the unit you're using is a rectangle that has a length the size of the one stick and the width the size of the other stick, so you're sort of going backwards and ending up with a unit that means something…. (Simon & Blume, 1994b, p. 489).

Through the discussion of this problem and an extension question that asked them to express the area of Ruiz and Louisa's rectangular area in terms of other units, I was persuaded that most of the students understood the relationship between the linear measures and the area measures in this context. The next step was to see if they could use this understanding to revisit the problem involving turning the rectangle. Would the understandings that they had developed in the stick problem allow them to question their assumption of the cardboard rectangle as the appropriate unit of measure? The revised TR problem was an attempt to make the problem more concrete by having them actually measure out particular rectangular regions using the method in question.

> *Revised TR problem.* I used your [cardboard] rectangle and my method (rotating the rectangle) to measure two rectangular regions; one was 3×4 and the other was 5×2. Draw these regions (real size). Record all that you can determine about their areas.

Half of the small groups determined that squares were useful units for describing the areas of the rectangular regions and could explain their thinking.

Eve: First, when we started drawing it, we drew like all the rectangles, OK? So it showed the overlapping up in the corner, but then we thought just take away the overlapping …, and just think of it as a side, like this is a side and this is a side, like the sticks that Louisa and Ruiz used, just to get this little edge right here as a stick and this little edge right here as a stick, OK?… So what it actually makes is squares, because they're the same length … you wouldn't talk about your rectangles, 'cause it has no relevance. (Simon & Blume, 1994b, p. 490–491)

Some of the students who had not previously seen the usefulness of square units in the revised TR problem did so in the course of the class discussion. However, a number of students, who saw that measuring in squares worked for the problem, were unclear how one would know "when to use rectangles and when to use squares."

DISCUSSION

The discussion of the teaching situations is divided into two parts. The first part examines the teacher's role that emerges in terms of the decision making about content and task. This discussion, which focuses on the composite picture of teaching seen across the three situations presented, leads to the articulation of a model of teacher decision making called the Mathematics Teaching Cycle. The second part of the discussion highlights particular aspects of each of the situations (considered separately), in an attempt to further elaborate the role of the teacher.

As teacher/researcher, I began the mathematics course with particular theoretical perspectives on teaching and learning, some of which were articulated earlier in this paper. However, not surprisingly, these perspectives did not "tell" me what to do as challenges arose. My responses to these challenges often were not the result of well-articulated models. Rather, they were emerging patterns of operation in problem situations. It is only through a posteriori analysis that these patterns have been characterized and used to enhance, modify, or develop theory. Thus, the reader should keep in mind that the theoretical aspects of this section grew out of the data analysis and were not necessarily part of my explicit thinking as I was planning and teaching. Also, these patterns of operation were established over repeated encounters with similar problem situations. I had taught essentially this way for many years. This particular teaching experience led to further elaboration of my teaching; the teaching-experiment design led to a new level of analysis of the teaching.

In this section, I use the first person singular to refer to my actions and thinking as the teacher. I use the third person, often referring to "the teacher" to designate ideas that I am lifting from the particular context in which I was the teacher. Because I was the teacher in these episodes, I use male pronouns in this section when referring to the teacher generically.

Unpacking the Teaching Episodes: Developing Theory

This section analyzes the teacher's role as decision maker as it emerges across the three teaching situations.

The rectangles lesson was shaped by my understanding of the multiplicative relationship between the area of a rectangle and its linear measures. (The focus on my personal knowledge does not discount that this knowledge can be viewed as socially constituted and taken-as-shared in the mathematics education community. Rather, it is expedient here to focus on my particular interpretation of these socially accepted ideas.) My previous experience with prospective elementary teachers led me to hypothesize that my students would not share this knowledge. Rather, I expected that their knowledge would be rule bound and that the concepts underlying the formula for the area of a rectangle would be unexplored. The disparity between my understanding, which I judged to be useful, and my sense of their understanding defined my learning goal for the first segment.

Note that hypotheses of students' understandings may be based on information from a variety of sources: experience with the students in the conceptual area, experience with them in a related area, pretesting, experience with a similar group, and research data. Initial hypotheses often lack data that are available as work with the students proceeds. Thus, the hypotheses are expected to improve (i.e., become more useful).

Having established my initial goal, that students would understand the relationship of multiplying length by width to the evaluation of the area of a rectangle, I considered possible learning activities and the types of thinking and learning that they might provoke. Following is a partial reconstruction of my thought process.

> I suspect that for many of my students $A = l \times w$ is a formula that has no conceptual roots. Concrete experience with area might be helpful. I need to keep in mind that they

> come to the task already knowing the rote formula. A learning situation that does not look like their previous experiences with area might preempt their resorting immediately to rote procedures.
>
> Tiling a rectangular region would provide the concrete experience. If I can envision a situation in which they form multiple units, sets of tiles, they may see the appropriateness of multiplication, in essence "deriving" the formula. I will not mention area but just ask them to find out how many tiles. However, if they are to make connections with $l \times w$, they must do more than count each tile.
>
> If I give them only one small tile, they will need to look for an efficient way of determining the number of tiles—which will encourage them to go beyond counting all of the tiles. If I use rectangular tiles, they will not be able to measure mindlessly; they will need to consider how their measuring relates to the placement of the tiles on the table. Measuring with a nonsquare rectangle to determine the area encourages a level of visualization that is not required when one uses a ruler to determine square units, that is, they will have to take into account *what* they are counting, the unit of measure, which is based on *how* they are laying the tiles on the table.

The preceding thought process provides an example of the reflexive relationship between the teacher's design of activities and consideration of the thinking that students might engage in as they participate in those activities. The consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage make up the *hypothetical learning trajectory*, a key part of the Mathematical Learning Cycle described in the next section.

Besides the teacher's knowledge of mathematics and his hypotheses about the students' understandings, several areas of teacher knowledge come into play, including the teacher's theories about mathematics teaching and learning; knowledge of learning with respect to the particular mathematical content (deriving from the research literature and/or the teacher's own experience with learners); and knowledge of mathematical representations, materials, and activities. The Mathematical Learning Cycle portrays the relationship of these areas of knowledge to the design of instruction.

The only thing that is predictable in teaching is that classroom activities will not go as predicted. Although the teacher creates an initial goal and plan for instruction, it generally must be modified many times (perhaps continually) during the study of a particular conceptual area. As students begin to engage in the planned activities, the teacher communicates with and observes the students, which leads the teacher to new understandings of the students' conceptions. The learning environment evolves as a result of interaction among the teacher and students as they engage in the mathematical content. Steffe (1990) points out, "A particular modification of a mathematical concept cannot be caused by a teacher any more than nutriments can cause plants to grow" (p. 392). A teacher may pose a task. However, it is what the students make of that task and their experience with it that determines the potential for learning.

Student responses to the rectangle problems led me to believe that students did not adequately understand what is meant by area. As a result, I generated a new learning goal, understanding area. This goal temporarily superseded but did not replace the original learning goal. Toward this end I posed the blob problem, anticipating that the students would brainstorm some ways to find area, discuss those ways, and in so doing strengthen their understanding of area. However, the specifics of what happened resulted in additional, unanticipated learning. First, students proposed the string strategy. On the basis of the understanding that I had developed of the students' conceptual difficulties in considering the string strategy, and on the basis of the interesting contrast that I saw between the string strategy and the dough strategy (as a result of my own mathematical understandings), I revised my goal for instruction once again. I now saw as the (local) goal facilitating students' understanding of conservation of area (not limited to Piaget's assessment of the concept with children). I intended for my students to deal with the question, "Under what types of change in shape does the area of a region remain invariant?"

My interest in their constructing answers to this question was based on three factors: (a) I believed that it would further their understanding of area, my motivation for posing the blob problem; (b) I saw an opportunity for learning based on the juxtaposition of the two strategies—an opportunity that I had neither planned for nor anticipated; and (c) I believed that the concept of invariance, which I had thought about previously in relation to arithmetic concepts but not area, was an important one. The third factor also points out how my own understanding of the mathematical connections involved is enhanced as I attend to the mathematical thinking of my students. This evolution of the teacher's mathematical knowledge is also revealed in the analysis of the third episode, the data involving measuring with only the long side of the rectangle (TR problem).

My original goal that motivated the rectangles lesson was for my students to understand the evaluation of the area of a rectangle as a multiplicative relationship between the linear measures of the sides. For me, as I began instruction, such an understanding involved connecting an understanding of multiplication-as-repeated-addition with the notion of identical rows of units of area and understanding the relationship between linear units and area units. The latter concept was represented by the issue of turning the rectangle to measure—I had not unpacked this understanding further.

The classroom discussion, however, pushed me to reexamine these understandings and to further elaborate my map of the conceptual terrain. (The use of the term "map" in this context is meant to emphasize that the teacher's understandings serve as a map as he engages in making sense of students' understandings and identifies potential learnings.) The students' misanticipation of the area unit (assumption that the area would necessarily be measured in terms of cardboard rectangles) led me to explore the importance of anticipating an appropriate unit. Anticipating the area unit seemed to involve both an anticipation of the organization of the units, a rectangular array, and an understanding that the linear units define the size and shape of the units within that array. (For a fuller discussion, see Simon & Blume, 1994b.) The multiplicative relationship, therefore, involved the coordination of the linear units to determine an area unit within an anticipated rectangular array. What I had observed in my students had changed both my perspective on my students' knowledge and my perspective on the mathematical concepts involved (my internal map). This reorganization of my perspectives led to a modification of my goals, my plans for learning activities, and the students' learning and thinking that I anticipated.

The Mathematics Teaching Cycle

The analysis of these teaching episodes has led to the development of the Mathematics Teaching Cycle (Figure 4) as a schematic model of the cyclical interrelationship of aspects of teacher knowledge, thinking, decision making, and activity that seems to be demonstrated by the data.

The three episodes create a picture of a teacher whose teaching is directed by his conceptual goals for his students, goals that are constantly being modified. The original lesson involving the rectangles on the table was not a random choice, nor was it Chapter 1 in someone's textbook. The goal for and design of the lesson were based on relating two factors: the teacher's mathematical understanding and the teacher's hypotheses about the students' knowledge. I refer to "hypotheses" about students knowledge to emphasize that the teacher has no direct access to students' knowledge. He must infer the nature of the students' understandings from his interpretations of his students' behaviors, based on his own schemata with respect to mathematics, learning, students, and so on. It is implied that the teacher can compare his understanding of a particular concept to *his* construction of the students' understandings, not to the students' "actual" understandings.

As the teacher, my perception of students' mathematical understandings is structured by my understandings of the mathematics in question. Conversely, what I observe in the students' mathematical thinking affects my understanding of the mathematical ideas involved and their interconnections. These two factors are interactive spheres of a teacher's thinking (Ball's, 1993, "bifocal perspective" discussed earlier).

Steffe (1990) states,

> Using their own mathematical knowledge, mathematics teachers must interpret the language and actions of their students and then make decisions about possible mathematical knowledge their students might learn. (p. 395)

The teacher's learning goal provides a direction for a *hypothetical learning trajectory*.[6] I use the term "hypothetical learning trajectory" to refer to the teacher's prediction as to the path by which learning might proceed. It is hypothetical because the actual learning trajectory is not knowable in advance. It characterizes an expected tendency. Individual students' learning proceeds along idiosyncratic, although often similar, paths. This assumes that an individual's learning has some regularity to it (cf. Steffe, et al., 1983, p. 118), that the classroom community constrains mathematical activity often in predictable ways, and that many of the students in the same class can benefit from the same mathematical task. A hypothetical learning trajectory provides the teacher with a rationale for choosing a particular instructional design; thus, I make my design decisions based on my best guess of how learning might proceed. This can be seen in the thinking and planning that preceded my instructional interventions in each of the teaching situations described as well as the spontaneous decisions that I made in response to students' thinking.

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[6]I choose to use "hypothetical learning trajectory," rather than traditional terminology, to emphasize aspects of teacher thinking that are grounded in a constructivist perspective and that are common to both advanced planning and spontaneous decision making.

The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. The creation and ongoing modification of the hypothetical learning trajectory is the central piece of the model that is diagrammed in Figure 4. The notion of a hypothetical learning trajectory is not meant to suggest that the teacher always pursues one goal at a time or that only one trajectory is considered. Rather, it is meant to underscore the importance of having a goal and rationale for teaching decisions and the hypothetical nature of such thinking. Note that the development of a hypothetical learning process and the development of the learning activities have a symbiotic relationship; the generation of ideas for learning activities is dependent on the teacher's hypotheses about the development of students' thinking and learning; further generation of hypotheses of student conceptual development depends on the nature of anticipated activities.

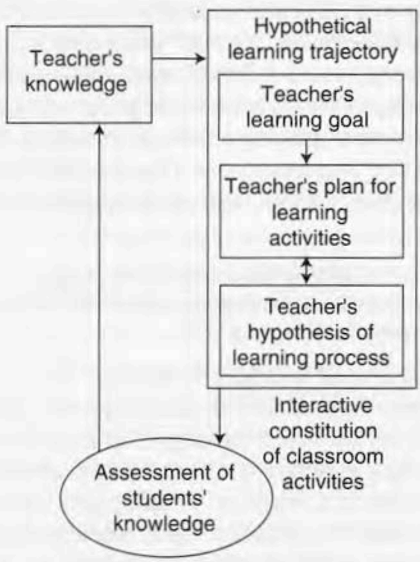

Figure 4. Mathematics teaching cycle (abbreviated).

The choice of the word "trajectory" is meant to refer to a path, the nature of which can perhaps be clarified by the following analogy. Consider that you have decided to sail around the world in order to visit places that you have never seen. One does not do this randomly (e.g., go to France, then Hawaii, then England), but neither is there one set itinerary to follow. Rather, you acquire as much knowledge relevant to planning your journey as possible. You then make a plan. You may initially plan the whole trip or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions with people along the way. You add destinations that prior to your trip were unknown to you. The path that you travel is your "trajectory." The path that you anticipate at any point in time is your "hypothetical trajectory."

The generation of a hypothetical learning trajectory prior to classroom instruction is the process by which (according to this model) the teacher develops a plan for classroom activity. However, as the teacher interacts with and observes the students, the teacher and students collectively constitute an experience. This experience by the nature of its social constitution is different from the one anticipated by the teacher. Simultaneous with and in interaction with the social constitution of classroom activity is a modification in the teacher's ideas and knowledge as he makes sense of what is happening and what has happened in the classroom. The diagram in Figure 4 indicates that the assessment of student thinking (which goes on continually in the teaching model presented) can bring about adaptations in the teacher's knowledge that, in turn, lead to a new or modified hypothetical learning trajectory.

Figure 5 describes the relationship among various domains of teacher knowledge, the hypothetical learning trajectory, and the interactions with students.

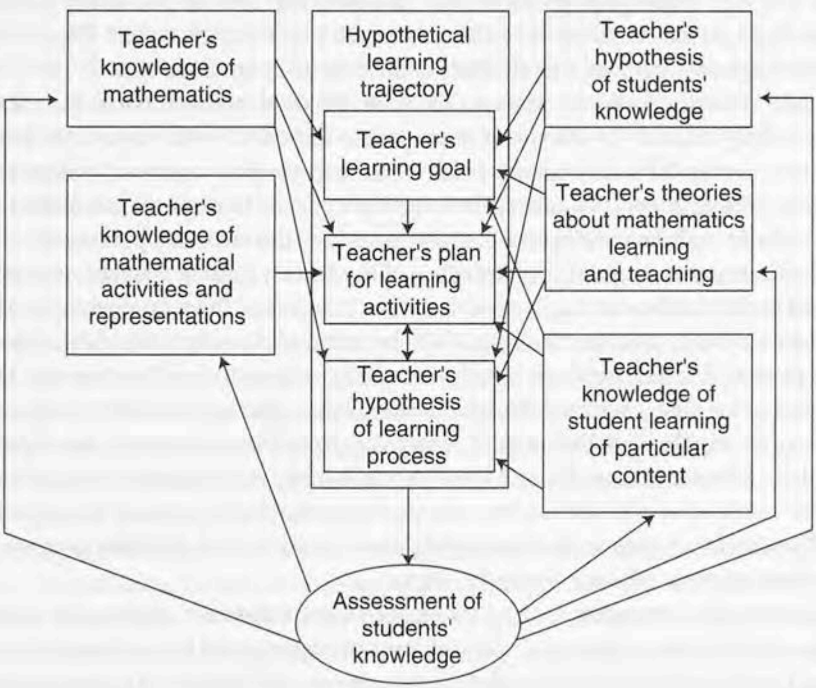

Figure 5. Mathematics teaching cycle. (The domains of teacher knowledge also inform "assessment of students' knowledge" directly. However, because this was not the emphasis of the model, and in order to simplify the diagram, those arrows are not included.)

Beginning at the top of the diagram, the teacher's knowledge of mathematics in interaction with the teacher's hypotheses about the students' mathematical knowledge contribute to the identification of a learning goal. These domains of knowledge, the learning goal, and the teacher's knowledge of mathematical activities and representation, his knowledge of students' learning of particular content, as well as the teacher's conceptions of learning and teaching (both within mathematics and in general) contribute to the development of learning activities and a hypothetical learning process.

The modification of the hypothetical learning trajectory is not something that only occurs during planning between classes. The teacher is continually engaged in adjusting the learning trajectory that he has hypothesized to better reflect his enhanced knowledge. Sometimes fine tuning is in order, while at other times the whole thrust of the lesson must be discarded in favor of a more appropriate one. Regardless of the extent of modification, changes may be made at any or all of the three components of the hypothetical learning trajectory: the goal, the activities, or the hypothetical learning process.

Other Aspects of the Teacher's Role

Each of the three teaching situations portrays particular aspects of what teaching, which embodies reform principles, might be like. I discuss a few of these in this section.

The original rectangles problem was planned for one or two class periods; instead, eight periods were spent on the mathematics that was generated. Experienced teachers might affirm that it is difficult to determine in advance exactly how long it will take to teach a particular concept. However, the discrepancy between the amount of time anticipated and the amount of time spent in this case is well beyond the imprecision of planning. This discrepancy points at the experimental nature of mathematics teaching. "Experimental" denotes the ongoing cycle of hypothesis generation (or modification) and data collection that characterizes the teaching portrayed.

In the first situation, involving the tiling of the tables, I, as the teacher, perceived a lack of understanding among a majority of the students of the relationship between length-times-width and the counting of all the rectangles on the table. My response was to pose additional problems based on students' conceptual difficulties that I had witnessed in the past. I selected thought processes that I thought students could determine as not viable, but that would likely be problematic initially for them to invalidate. (Rectangles problem 2 is an example.) My rationale was that previous students' conceptual difficulties (from the teacher/researcher's perspective) are potential difficulties for my current students and represent useful hurdles for them to encounter in the development of more powerful ideas.

This approach represents a sharp contrast to the approach to instruction characteristic of traditional mathematics instruction and represented by mathematics textbooks. Traditional instruction tends to focus on one skill or idea at a time and then provide considerable routine practice to "reinforce" that learning. The mathematics is subdivided into small segments for instruction so that students can experience success on a regular basis. In contrast, Situation 1 demonstrates a view of learning as one involving a complex network of connections. Learning is likely to be fostered by challenging the learner's conceptions using a variety of contexts. The teacher can be compared to an athletic coach who employs a variety of practice activities that challenge the athletes' strength and skill, often beyond what is required of the athlete in competition (dribbling two basketballs while blindfolded, playing a soccer game where each player may not touch the ball two consecutive times, performing a figure skating program three times in a row with only a 2-minute rest in between). These activities are not aimed at constant success, but rather at increased competence. Growth is a result of challenge to body and mind. Conceptual difficulties that I have previously observed in students are not to be avoided; rather, they provide particular challenges, which if surmounted by the students, result in conceptual growth. This fits with French researchers' notion of "obstacles epistemologiques" (Bachelard, 1938, cited in Brousseau, 1983), that overcoming certain obstacles is a natural and essential part of conceptual development. These obstacles are a result of prior concepts that, although adaptive in earlier contexts, are maladaptive given the demands of the current problem situation.

A second feature of the approach seen in Situation 1 is subtler. As a teacher, I often do not have a well-developed map of the mathematical conceptual area in which I am engaging my students; that is, I may not have fully articulated for myself (or found in the literature) the specific connections that constitute understanding or the nature of development of understanding in that area. Rather, as was the case when I started the rectangles instructional unit, my knowledge of what it means to understand the particular concept may be carried in part by particular problem situations. The kinds of difficulties that students encounter provide me with key pieces of what it means to understand. Thus, in such cases, my operational definition of understanding is the ability to overcome these particular difficulties; I may not have unpacked the difficulties in order to understand the conceptual issues that are implicated. Thus, even if I do not have a thorough knowledge of what constitutes mathematical understanding in a particular domain, having a rich set of problem situations that challenge students and having knowledge of conceptual difficulties that they typically encounter provide me with an approximation that lets me be reasonably effective in promoting learning in the absence of more elaborated knowledge. (This is not to suggest that the more elaborated understanding would not be more powerful.) Indeed, engaging students in these problem situations and with these conceptual difficulties gives me an opportunity to learn more about what it means to understand the concepts involved.

Underlying Situation 2 is an idea that highlights a difference between teaching based on a traditional view of learning and teaching based on a constructivist perspective. Rather than "review" what is meant by area or assign "practice problems," my approach was to challenge the students in a way that might push them to extend their conceptions of area. The review-and-practice approach is based on learning as improving storage and retrieval of received information. (Although I am not negating the importance of memory, I contend that it is not what is most important, most interesting, and most problematic for educators in the domain of mathematics.) My approach in Situation 2 reflected a view of understanding as a network of connected ideas that is further elaborated as the understanding is used to solve novel problems.

Situation 3 was the most difficult to analyze. I brought to the teaching situations a view that learning is triggered by disequilibrium. When the students were convinced that the rotated rectangle method of measuring and calculating provided no useful information about the table, I tried in every way I could to provoke disequilibrium, but to no avail. In-depth analysis of the data suggests that my interpretations of the students' thinking that led to their conclusion was not adequate. Whereas I had thought that they saw my method as counting the number of overlapping rectangles, I now believe that they were saying that the method involving turning the rectangle counted nothing because in the process I was overlapping the rectangles. This subtle difference in thinking may account for my inability to foster disequilibrium.

Having failed to promote disequilibrium, I embarked intuitively on another strategy. I backed away from the particular problem to try to focus on a part of the understanding demanded by the problem. A posteriori analysis suggests that what I was doing was fostering the development of knowledge that, when the students returned to that problem, might contribute to the students' experiencing cognitive conflict. In this case, if I could help students build an understanding of the relationship between linear and area measures of a rectangle, they would then experience a conflict between those understandings and the expectation that measuring with the cardboard rectangle resulted in a measurement where that rectangle was the unit of area. This teaching episode seems to emphasize that disequilibrium is not created by the teacher. He can try to promote disequilibrium. However, the success of such efforts is in part determined by the adequacy of his model of students' understanding. It also seems to support the notion that learning does not proceed linearly. Rather, there seem to be multiple sites in one's web of understandings on which learning can build.

IN SUMMARY

Constructivist views of learning have provided a theoretical foundation for mathematics education research and a framework within which teachers can understand their students. However, constructivism also poses a challenge to the mathematics education community to develop models of teaching that build on, and are consistent with, this theoretical perspective. Small-group interaction, nonroutine problem solving, and manipulative materials can be valuable tools in the hands of mathematics teachers. Yet the ability to use these tools is not sufficient to allow teachers to be the architects of productive learning situations resulting in conceptual growth. Theoretically based frameworks for teaching have the potential to guide the use of these tools.

By what means can a teacher help students to develop new, more powerful mathematical concepts? Novice teachers, who want their students to "construct" a particular idea, often ask for the idea from their students, consciously or unconsciously hoping that at least one student will be able to explain it to the others (Simon, 1991). Such an approach does not deal with a key question: If a group of students do not have a particular concept, how does a teacher work with them to foster their development of that concept?

The principal currencies of the mathematics teacher (if lecturing is rejected as an effective means of promoting concept development) are the posing of problems or tasks and the encouragement of reflection. The data analysis described in this paper and the resulting Mathematics Teaching Cycle address the issue of the process by which a teacher can make decisions as to the content, design, and sequence of mathematical tasks. The model emphasizes the important interplay between the teacher's plans and the teacher and students' collective constitution of classroom activities. The former involves creation of instructional goals and hypotheses about how students might move towards those goals as a result of their collective engagement in particular mathematical tasks. However, the teacher's goals, hypotheses about learning, and design of activities change continually as the teacher's own knowledge changes as a result of being involved in the culture of the mathematics classroom.

> A goal structure for mathematics education such as the one elaborated by Treffers (1987) is needed in specifying possible learning environments by teachers. But this element of possible learning environments is just as dependent on the experiential fields that constitute learning environments as the latter are dependent on the former. Mathematics educators should not take their goals for mathematics education as fixed ideals that stand uninfluenced by their teaching experiences. Goal structures that are established prior to experience are only starting points and must undergo experiential transformation in actual learning and teaching episodes. (Steffe, 1991, p. 192)

Steffe's comments seem to underscore the cyclical nature of this teaching process.

The Mathematics Teaching Cycle portrays a view of teacher decision making with respect to content and tasks that has been shaped by the meeting of a social constructivist perspective with the challenges of the mathematics classroom. Several themes are particularly important in the approach to decision making represented by this model.

1. Students' thinking and understanding is taken seriously and given a central place in the design and implementation of instruction (consistent with Steffe, 1991). Understanding students' thinking is a continual process of data collection and hypothesis generation.

2. The teacher's knowledge evolves simultaneously with the growth in the students' knowledge. As the students are learning mathematics, the teacher is learning about mathematics, learning, teaching, and about the mathematical thinking of his students.

3. Planning for instruction is seen as including the generation of a hypothetical learning trajectory. This view acknowledges and values the goals of the teacher for instruction and the importance of hypotheses about students' learning processes (ideas that I hope I have demonstrated are not in conflict with constructivism).

4. The continually changing knowledge of the teacher (see #2) creates continual change in the teacher's hypothetical learning trajectory.

These last two points address directly the question raised earlier in the paper of balance between direction (some may call this "structure") and responsiveness to students, a creative tension that shapes mathematics teaching. The model suggests that, as mathematics teachers, we strive to be purposeful in our planning and actions, yet flexible in our goals and expectations.

The mathematics education literature is strong on the importance of listening to students and assessing their understanding. However, the emphasis on anticipating students' learning processes is not developed by most current descriptions of reform in mathematics teaching. Research on how students develop particular mathematical knowledge (cf. Steffe, et al., 1983, see pp. 118 & 135; Thompson, 1994) informs such anticipation. Perhaps one explanation for the success of Cognitively Guided Instruction (Carpenter, Fennema, Peterson, & Carey, 1988), in which teachers learned about research on children's thinking (Carpenter & Moser, 1983), is that it increased teachers' ability to anticipate children's learning processes. Ball (1993) articulates a similar position emphasizing the role of teachers' hypotheses about student learning. "Selection of representational contexts involves conjectures about teaching and learning, founded on the evolving insights about the children's thinking and [the teacher's] deepening understanding of the mathematics" (p. 166).

The data from this study must be seen in its particular context. The teaching practice was embedded in a teacher education program; the mathematics students were prospective elementary students. As the teacher, I felt no pressure to teach from a preset curriculum nor to cover particular mathematical content, a condition that is probably the exception rather than the rule for mathematics teachers. Mathematics teaching with other populations involves a set of different constraints. Research in other contexts will inform us about the degree of context dependence of the ideas generated.

A possible contribution that can be made by the analysis of data and the resulting model reported on in this paper is to encourage other researchers to examine teachers' "theorems in action" and to make teachers' assumptions, beliefs, and emerging theories about teaching explicit. At the minimum, the paper should serve to emphasize the need for models of mathematics teaching that are consistent with, and built on, emerging theories of learning. Much research remains to be done to understand the implications for practice of teachers holding constructivist perspectives. It is the recognition that constructivism does not tell us how to teach that will motivate increased work in this area.

A well-developed conception of mathematics teaching is as vital to mathematics teacher educators as well-developed conceptions of mathematics are to mathematics teachers. Informed decisions in each case are dependent on a clear sense of the nature of the content being taught. Considering the Mathematics Teaching Cycle as a way to think about mathematics teaching means that teachers would need to develop abilities beyond those already currently focused on in mathematics education reform, particularly the ability to generate hypotheses about students' understandings (which goes beyond soliciting and attending to students' thinking), the ability to generate hypothetical learning trajectories, and the ability to engage in conceptual analysis related to the mathematics that they teach. This last point supports proposed reforms of mathematics for teachers (cf. Cipra, 1992; Committee on the Mathematical Education of Teachers, 1991) and supports arguments that the mathematical preparation of teachers is far from adequate if teachers are to engage in pedagogical deliberations as characterized in this paper.

Finally, it should be noted that the role of the mathematics teacher as portrayed in this paper is a very demanding one. Teachers will need access to relevant research on children's mathematical thinking, innovative curriculum materials, and ongoing professional support in order to meet the demands of this role.

REFERENCES

Bachelard, G. (1938). *La formation de l'esprit scientifique* [The development of the scientific mind—my translation]. Paris: VRIN, 1975.

Balacheff, N. (1990). Towards a problematique for research on mathematics teaching. *Journal for Research in Mathematics Education, 21*, 258–272.

Ball, D. L. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching: Vol. 2, Teachers' subject matter knowledge and classroom instruction* (pp. 1–48). Greenwich, CT: JAI Press.

Ball, D. L. (1993). Halves, pieces, and twoths: Constructing representational contexts in teaching fractions. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157–196). Hillsdale, NJ: Lawrence Erlbaum.

Bauersfeld, H. (1995). Development and function of mathematizing as a social practice. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 137–158). Hillsdale, NJ: Lawrence Erlbaum.

Blumer, H. (1969). *Symbolic interactionism*. Englewood Cliffs, NJ: Prentice Hall.

Brousseau, G. (1981). Problèmes de didactique des décimaux [Problems in teaching decimals]. *Recherches en Didactiques des Mathématiques, 2.1* 37–125.

Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques [Epistemological obstacles and mathematical problems—my translation]. *Recherches en Didactiques des Mathématiques, 4.2*, 164–198.

Brousseau, G. (1987). *Les différents rôles du maître* [The different roles of the teacher—my translation]. Colloquium of P.E.N. Angers (unpublished).

Carpenter, T., Fennema, E., Peterson, P., & Carey, D. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education, 19*, 385–401.

Carpenter, T. & Moser, J. (1983). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*, 179–202.

Cipra, B. (Ed.) (1992). *On the mathematical preparation of elementary school teachers*. Report of a two-part conference held at the University of Chicago in January and May, 1991.

Cobb, P. (1989). Experiential, cognitive, and anthropological perspectives in mathematics education. *For the Learning of Mathematics, 9*, 32–42.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14*, 83–94.

Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In E. Forman, N. Minick, & A. Stone (Eds.), *Contexts for learning: Sociocultural dynamics in children's development* (pp. 91–119). New York: Oxford University Press.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D. McLeod & V. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 117–148). New York: Springer-Verlag.

Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics, 23*, 99–122.

Cobb, P., Yackel, E., & Wood, T. (1993). Learning mathematics: Multiple perspectives, theoretical orientation. In T. Wood, P. Cobb, E. Yackel, & D. Dillon, (Eds.), *Rethinking elementary school mathematics: Insights and issues. Journal for Research in Mathematics Education* Monograph Series, Number 6, (pp. 21–32). Reston, VA: National Council of Teachers of Mathematics.

Committee on the Mathematical Education of Teachers (1991). *A call for change: Recommendations for the mathematical preparation of teachers of mathematics*. Washington, DC: Mathematical Association of America.

Confrey, J. (1995). The relationship between radical constructivism and social constructivism. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 185–226). Hillsdale, NJ: Lawrence Erlbaum.

Douady, R. (1985). The interplay between different settings, tool-object dialectic in the extension of mathematical ability. In L. Streefland (Ed.), *Proceedings of the ninth international conference for the Psychology of Mathematics Education*, Vol. II, 33–52, Utrecht, The Netherlands.

Hart, K. M. (1981). *Children's understanding of mathematics: 11–16*. London: John Murray.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence*. New York: Basic Books.

Karplus, R., Karplus, E., Formisano, M. & Paulsen, A. (1979). Proportional reasoning and control of variables in seven countries. In J. Lochhead and J. Clement (Eds.), *Cognitive process instruction* (pp. 47–103). Hillsdale, NJ: Lawrence Erlbaum.

Konold, C. (1995). Social and cultural dimensions of knowledge and classroom teaching. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 175–184). Hillsdale, NJ: Lawrence Erlbaum.

Laborde, C. (1989). Hardiesse et raison des recherches françaises en didactique des mathématiques [The audacity and reason of French research in mathematics education]. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Actes de la 13ème conference internationale de Psychology of Mathematics Education*, Vol. I, 46–61, Paris, France.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*, 29–63.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Piaget, J. (1970). *Genetic epistemology*. New York: Columbia University Press.

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 13–51). Dordrecht, The Netherlands: Kluwer.

Simon, M. (1991). *Report of classroom observations: Evaluation of the Mathematics Leadership Network*. Unpublished manuscript available from the author.

Simon, M., & Blume, G. (1994a). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, *13* (2).

Simon, M., & Blume, G. (1994b). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education*, *25*, 472–494.

Simon, M., & Blume, G. (in press). Mathematical justification in the classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior*.

Steffe, L. (1990). Mathematics curriculum design: A constructivist perspective. In L. Steffe & T. Wood (Eds.), *Transforming children's mathematics education: International perspectives*, (pp. 389–398). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. (1991). The constructivist teaching experiment: Illustrations and implications. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education*, (pp. 177–194). Dordrecht, The Netherlands: Kluwer.

Steffe, L., & Gale, J. (Eds.) (1995). *Constructivism in education*. Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York: Praeger Scientific.

Steffe, L., & Wiegel, H. (1992). On reforming practice in mathematics education. *Educational Studies in Mathematics*, *23*, 445–465.

Streeck, J. (1979). Sandwich. Good for you. In J. Dittman (Ed.), *Arbeiten zur konversations analyse* (pp. 235–257). Tubingen, Germany: Niemeyer.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction—The Wiskobas Project*. Dordrecht, The Netherlands: Reidel.

von Glasersfeld, E. (1987). Learning as a constructive activity. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*, (pp. 3–17). Hillsdale, NJ: Lawrence Erlbaum.

von Glasersfeld, E. (1991). *Radical constructivism in mathematics education*. Dordrecht, The Netherlands: Kluwer.

von Glasersfeld, E. (1995). A constructivist approach to teaching. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 3–16). Hillsdale, NJ: Lawrence Erlbaum.

Wertsch, J., & Toma, C. (1995). Discourse and learning in the classroom: A sociocultural approach. In L. Steffe & J. Gale (Eds.), *Constructivism in education*. Hillsdale, NJ: Lawrence Erlbaum.

Wood, T., Cobb, P., & Yackel, E. (1995). Reflections on learning and teaching mathematics in elementary school. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 401–422). Hillsdale, NJ: Lawrence Erlbaum.

Wood, T., Cobb, P., Yackel, E., & Dillon, D., (Eds.) (1993). *Rethinking elementary school mathematics: Insights and issues. Journal for Research in Mathematics Education* Monograph Series, Number 6. Reston, VA: National Council of Teachers of Mathematics.

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